Section 2.6

Continuous Functions

The idea of continuity has a straightforward graphical interpretation: a function is *continuous* at $x = c$ if the graph has no hole or break at c. In particular $f(c)$ must be defined.

Example 2.6.1. Consider if the following function is continuous at $x = 0$ (graph)

$$h(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Definition 2.6.2 (Limit Definition of Continuity)**.** A function f is *continuous* at c if $\lim_{x \to c} f(x) = f(c)$.

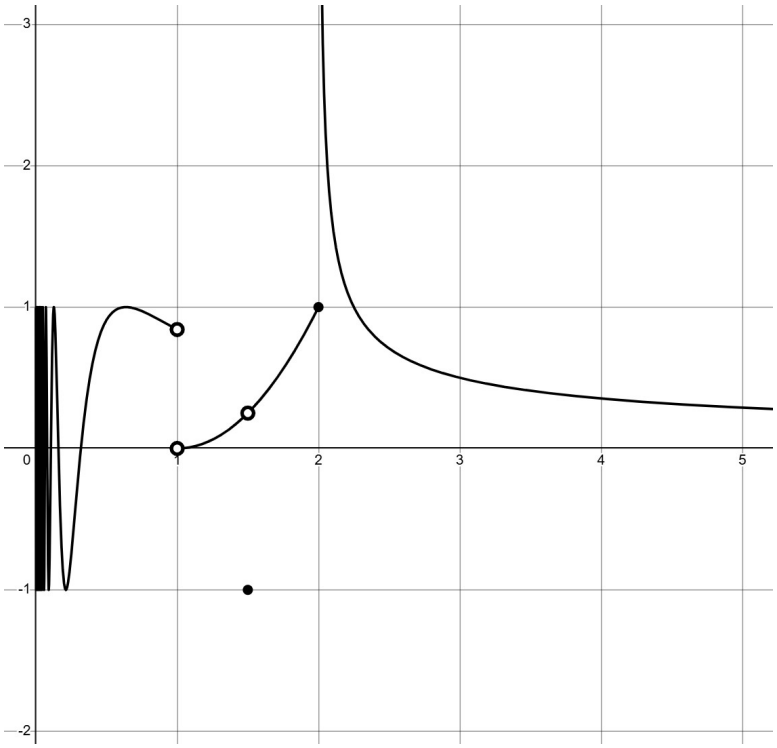
Cheat Sheet 2.6.3. To check if f is continuous at c:

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(a)$

Back to $h(x)$ above, is h continuous at 0?

If f is not continuous at c we say f has a *discontinuity* at c.

Types of Discontinuities



Example 2.6.4. Determine if the function is continuous at the given point.

1. $f(x) = 2x^3 - 3x + 4$ at $x = 2$

2. $g(x) = \dfrac{3x^2 + 2x + 1}{x - 1}$ at $x = 1$

3. $g(x) = \dfrac{3x^2 + 2x + 1}{x - 1}$ at $x = 2$

Theorem 2.6.5.

1. A polynomial is continuous at all values of x.
2. A rational function $f(x) = \dfrac{p(x)}{q(x)}$ is continuous at all values x such that $q(x) \neq 0$.

Theorem 2.6.6 (Combining Continuous Functions)**.** If f and g are continuous at c, k is some constant and $n > 0$ is an integer, then

1. $f + g$ is continuous at c.
2. $f - g$ is continuous at c.
3. kf is continuous at c.
4. fg is continuous at c.
5. $\frac{f}{g}$ is continuous at c provided $g(c) \neq 0$.
6. $(f(x))^n$ is continuous at c.
7. $f^{-1}(x)$ (if this function exists) is continuous at $f(c)$.

Theorem 2.6.7 (Continuity of Elementary Functions)**.** If f is an elementary function, then f is continuous at all values x in its domain. This includes:

1. Polynomials and rational functions,
2. Algebraic functions,
3. Trig and inverse-trig functions,
4. The functions b^x and $\log_b x$.

Theorem 2.6.8 (Compositions of Continuous Functions)**.** If g is continuous at c and f is continuous at $g(c)$ then $f \circ g$ is continuous at c.

Example 2.6.9. Determine the continuity of the following functions:

1. $\sin x^2$ at $x = 0$.

2. $e^{\cos^2 x}$ at $x = 3$.

We can strengthen the statement of Theorem 2.6.8:

Theorem 2.6.10. If $\lim_{x \to c} g(x) = L$ and f is continuous at L, then

$$\lim_{x \to c} f(g(x)) = f\big(\lim_{x \to c} g(x)\big)$$

Example 2.6.11. Compute

$$\lim_{x \to 2} \cos\left(\frac{x^2 - 4}{x - 2}\right)$$

Definition 2.6.12 (Left and Right Continuity)**.**
A function f is *right-continuous* at c if $\lim_{x \to c^+} f(x) = f(c)$.
A function f is *left-continuous* at c if $\lim_{x \to c^-} f(x) = f(c)$.

We can now define a concept we will use throughout the class:

Definition 2.6.13 (Continuity on an Interval)**.** For an interval I, we say the function f is *continuous on* I if it is continuous at all point of I. In particular, if a the left endpoint or b the right endpoint is included in I, then f is left-continuous at a or is right-continuous at b.

Example 2.6.14. Find the intervals of continuity for

$$f(x) = \begin{cases} x^3 + 4x + 1 & \text{if } x \leq 0 \\ 2x^3 & \text{if } x > 0 \end{cases}$$

Theorem 2.6.15 (Intermediate Value Theorem)**.** Suppose f is continuous on $[a, b]$ and L is a number strictly between $f(a)$ and $f(b)$. Then there exists a number c such that $a < c < b$ satisfying $f(c) = L$.

Example 2.6.16.

1. Determine if the equation $2^x + 5^x = 17^x$ has a solution for positive x.

2. Show that every cubic polynomial $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ has a real root (a solution to $p(x) = 0$).